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On static electrovacs with scalar fields

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Abstract. Static solutions of the Einstein–Maxwell scalar equations are obtained from known vacuum solutions, in which the specific electric charge as well as the specific scalar charge may assume arbitrary values. Two types of long-range scalar fields are considered: one is repulsive, like the Coulomb field, the other is attractive, like the gravity field. The effect of the superposition of these two types of fields is also studied. The solutions presented tend straightforwardly to the corresponding vacuum solutions when the constants associated with the electromagnetic and scalar strengths tend to zero.

1. Introduction

In a pioneering work Weyl (1917) obtained the class of solutions of Einstein–Maxwell equations for static systems with cylindrical symmetry, when there exists a functional relationship between the electrostatic potential ϕ and the metric component g_{00}. Later Majumdar (1947) generalized these results for electrostatic systems with or without spatial symmetry, showing that $g_{00} = 1+A\phi+\phi^2$; however, the interpretation of the constant A was not given. At the same time Papapetrou (1947) presented a special class of solutions involving magnetostatic potentials ϕ^* for which $g_{00} = (1+|\phi^*|)^2$. After that Bonnor (1954) showed how to generate a purely magnetostatic solution from a purely electrostatic one. Soon after, Ehlers (1955a) in a remarkable work presented a method for developing a purely electrostatic or a purely magnetostatic solution from a given vacuum solution; in the same year he extended (Ehlers 1955b) his results to include the constant A of Majumdar, and also introduced a one-parameter class of long-range scalar fields; however, in his solutions one finds difficulty in interpreting the constants associated with the electromagnetic and scalar fields, and also the original vacuum solutions cannot be obtained simply by putting the constants associated with these fields equal to zero. These difficulties are also encountered in later works of Bonnor (1961) and of Janis *et al* (1967). Other reeent approaches to the subject (but not including scalar fields) have been made by Harrison (1968), Geroch (1971) and Kinnersley (1973); Buchdahl (1959) considered long-range scalar fields, but did not include electromagnetic fields; and finally De (1969) studied the solutions of a set of equations of Einstein–Maxwell–Klein–Gordon combined fields, in which the inverse length parameter is associated with the mass of the system.

In this paper we present an extended solution of the problem proposed by Ehlers (1955a, b); using the operation of duality rotation we are able to introduce simultaneously electrostatic and magnetostatic fields. Differently from previous works, all the constants which appear in our solutions have simple interpretations at least in the weak-field approximation. Since scalar fields seem to play an interesting role in the structure of elementary particles (Teixeira *et al* 1975), we consider here simultaneously two possible classes of one-parameter scalar fields that can be fitted in Einstein's theory. An interesting feature of our solution is the straightforward recovery of the original vacuum solutions when one puts the constants associated with the strengths of the electromagnetic and scalar fields equal to zero.

2. Field equations

We consider the general static line element

$$ds^2 = e^{2\psi}(dx^0)^2 - e^{-2\psi}h_{ij}\,dx^i\,dx^j, \tag{1}$$

with ψ and h_{ij} functions of the space coordinates x^k only. Then the non-vanishing components of the Ricci tensor $R_{\mu\nu}$ are

$$R_{00} = -e^{4\psi}\Delta\psi, \tag{2}$$

$$R_{ij} = H_{ij} + 2\psi_{,i}\psi_{,j} - h_{ij}\Delta\psi, \tag{3}$$

where H_{ij} and Δ are the Ricci tensor and the Laplacian operator built up from h_{ij}, and a comma denotes ordinary derivative.

Maxwell's equations in empty space are

$$F^{\mu\nu}_{;\mu} = 0, \tag{4}$$

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu,\rho} = 0, \tag{5}$$

where a semicolon denotes covariant derivative and $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric unitary contravariant tensor density of weight $+1$; for stationary systems we can write, from (5)

$$F_{0i} = -\phi_{,i} \tag{6}$$

where from (4) and (1) the electrostatic potential ϕ satisfies

$$(h^{1/2}\,e^{-2\psi}h^{ij}\phi_{,i})_{,j} = 0, \tag{7}$$

with $h = \det h_{ij}$ and $h^{ij}h_{jk} = \delta^i_k$. The energy–momentum tensor

$$E_{\mu\nu} = (F_{\mu\alpha}F^{\alpha}{}_{\nu} - g_{\mu\nu}F_{\alpha\beta}F^{\beta\alpha}/4)/4\pi, \tag{8}$$

corresponding to an electromagnetic field whose only antisymmetric components are given by (6), has the surviving components

$$8\pi E_{00} = e^{2\psi}h^{ij}\phi_{,i}\phi_{,j}, \tag{9}$$

$$8\pi E_{ij} = (h_{ij}h^{km}\phi_{,k}\phi_{,m} - 2\phi_{,i}\phi_{,j})\,e^{-2\psi}. \tag{10}$$

Another long-range field which we will consider is the 'massless' real scalar field S which satisfies in regions free of scalar sources (Idel Wolk *et al* 1975) the equation $S^{;\mu}_{;\mu} = 0$, which in static systems (1) takes the form

$$(h^{1/2}h^{ij}S_{,i})_{,j} = 0. \tag{11}$$

There are indeed two classes of such fields: the first class is responsible for an attraction between two static sources of the same sign, so we call such gravity-like fields attractive: sources of opposite sign would repel each other. The second class is a Coulomb-like field, in the sense that two static sources of the same sign repel each other, while sources of opposite sign feel a mutual attraction; we call such fields repulsive. The energy–momentum tensor of a long-range scalar field is

$$S_{\mu\nu} = \pm(S_{,\mu}S_{,\nu} - g_{\mu\nu}S^{,\alpha}S_{,\alpha}/2)/4\pi \tag{12}$$

where the plus and minus signs correspond to attractive and repulsive scalar fields, respectively; the demonstration of this assertion is given in the appendix. Throughout this paper we follow this sign convention.

Let us consider now the Ehlers (1955b) problem: given a static solution (V, h_{ij}) of Einstein's vacuum equations

$$ds^2 = e^{2V}(dx^0)^2 - e^{-2V}h_{ij}\,dx^i\,dx^j, \tag{13}$$

$$R_{\mu\nu} = 0, \tag{14}$$

we want a static solution (ψ, h_{ij}, ϕ, S) of the Einstein–Maxwell scalar equations

$$ds^2 = e^{2\psi}(dx^0)^2 - e^{-2\psi}h_{ij}\,dx^i\,dx^j, \tag{15}$$

$$R_{\mu\nu} - g_{\mu\nu}R/2 = -8\pi(E_{\mu\nu} + S_{\mu\nu}), \tag{16}$$

with the electrostatic potential ϕ functionally related to the gravitational potential ψ, and also the scalar potential S functionally related to ψ; also this ψ is to bear a functional relationship with the vacuum gravitational potential V.

That is to say, given the functions V and h_{ij} of the space coordinates satisfying

$$(h^{1/2}h^{ij}V_{,i})_{,j} = 0, \tag{17}$$

$$H_{ij} + 2V_{,i}V_{,j} = 0, \tag{18}$$

we want ψ, ϕ and S satisfying Maxwell and scalar equations

$$(e^{-2\psi}h^{1/2}h^{ij}\phi_{,i})_{,j} = 0, \tag{19}$$

$$(h^{1/2}h^{ij}S_{,i})_{,j} = 0, \tag{20}$$

and also Einstein's equations

$$e^{2\psi}h^{-1/2}(h^{1/2}h^{ij}\psi_{,i})_{,j} = h^{ij}\phi_{,i}\phi_{,j}, \tag{21}$$

$$H_{ij} + 2\psi_{,i}\psi_{,j} - h^{-1/2}h_{ij}(h^{1/2}h^{km}\psi_{,k})_{,m} = -(h_{ij}h^{km}\phi_{,k}\phi_{,m} - 2\phi_{,i}\phi_{,j})\,e^{-2\psi} \mp 2S_{,i}S_{,j}, \tag{22}$$

each of the four functions (ψ, ϕ, S, V) having to be functionally related to the remaining three.

3. Solutions of the equations

From (17) and (19) and considering that both ϕ and ψ are functionally related with V, we see that

$$e^{-2\psi}\phi_{,i} = -aV_{,i}, \qquad a = \text{constant}; \tag{23}$$

similarly from (17) and (20) we get

$$S_{,i} = \pm cV_{,i}, \qquad c = \text{constant}. \tag{24}$$

Now substituting (18) and (21) into (22), and considering (23) and (24) gives us

$$V_{,i} = (C^2 + a^2 e^{2\psi})^{-1/2}\psi_{,i} \tag{25}$$

$$C = (1 \mp c^2)^{1/2} \tag{26}$$

which yields on integration

$$e^{-\psi} = \cosh CV - (1 + a^2/C^2)^{1/2} \sinh CV. \tag{27}$$

Then the electrostatic potential ϕ and the electrostatic field are, from (23), (27) and (6),

$$\phi = -(a/C)\, e^{\psi} \sinh CV, \tag{28}$$

$$F_{0i} = a\, e^{2\psi} V_{,i}, \tag{29}$$

while the scalar field is, from (24),

$$S = \pm cV. \tag{30}$$

4. Conclusions

In the case of vanishing electrostatic and scalar fields ($a = c = 0$), the two line elements (13) and (15) coincide, since from (27) and (26) we get $C = 1$ and $\psi = V$. This ease of recovery of the vacuum solution is an interesting feature of our solutions.

In the absence of scalar fields one gets from (27) and (28) Majumdar's relation

$$e^{2\psi} = 1 - 2(1 + a^2)^{1/2}\phi/a + \phi^2, \qquad c = 0; \tag{31}$$

for weakly charged systems (small values of a) we have from (28) the ratio ϕ/a finite.

We have imposed a functional relationship between the electrostatic potential ϕ and the gravitational potential ψ; in the weak-field approximation we have both $\psi \simeq 0$ and $V \simeq 0$, so from (28) $\phi \simeq -aV$. Except in some particular space-symmetric systems, the proportionality of these potentials can only be achieved when the ratio of the source of gravity to that of electrostatics is independent of position. Then the parameter a can be interpreted in this approximation as a constant ratio between the electrostatic charge density and the original mass density. Similarly the parameter c in (30) would be the constant ratio between a scalar source density and the original mass density.

It is known (Misner and Wheeler 1957) that the same energy–momentum tensor $E_{\mu\nu}$ corresponds to two different electromagnetic fields $F_{\mu\nu}$ and $F'_{\mu\nu}$ when these two fields are related by a duality rotation,

$$F'_{\mu\nu} = F_{\mu\nu} \cos\theta + {}^*F_{\mu\nu} \sin\theta, \tag{32}$$

where θ is an arbitrary real constant and ${}^*F_{\mu\nu}$ is the dual of $F_{\mu\nu}$,

$${}^*F^{\mu\nu} = (-g)^{-1/2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}/2; \tag{33}$$

as a consequence, we can generalize our solution (27) to

$$e^{-\psi} = \cosh CV - [1 + (a^2 + b^2)/C^2]^{1/2} \sinh CV \tag{34}$$

by including a magnetostatic field

$$F^{ij} = (b/a)(-g)^{-1/2}\epsilon^{0ijk}\phi_{,k} \tag{35}$$

where b is a constant related to the angle θ of the duality rotation by $\tan\theta = -b/a$; this field would be that produced by a magnetic monopole density having a constant ratio b to the original mass density, in the weak-field approximation.

From (26) and (27) one sees a fundamental difference between the introduction of an attractive scalar field and that of a repulsive one; the difference manifests itself in the square root $(1 \mp c^2)^{1/2}$. While a repulsive scalar field (lower sign) can be introduced with arbitrary strength c, the strength of an attractive scalar field is bounded to values of $c^2 < 1$. We can try a classical picture to see the origins of this difference. We can obtain a repulsive scalar field solution of the equations by introducing in the sources of the original static vacuum solution a certain amount of source of repulsive scalar field; in order to restore the equilibrium of the system we can introduce some additional mass (attractive); for strong repulsive scalar fields, large amounts of mass should be added. The situation is not the same, however, when one wants an attractive scalar field; if one starts from a static system and introduces in it an attractive scalar source, the static state can be maintained by taking off some of the original mass: since the mass of the original system is limited, one should expect a limit also on the amount of attractive scalar source which could be introduced without destroying the equilibrium.

One can have a superposition of a scalar field of the attractive class with one of the repulsive class; these are incoherent, and then the constant C in (27) is given by

$$C = (1 - c_a^2 + c_r^2)^{1/2}, \tag{36}$$

where c_a and c_r are the coefficients in (30) corresponding to the attractive and repulsive scalar fields, respectively.

One can still have a superposition of two or more scalar fields of the same class. That superposition can be either coherent or incoherent. A coherent superposition of two scalar fields of the same class occurs similarly to the superposition of two electrostatic potentials corresponding to two densities of electric charge, say; the net coefficient c in (30) of an attractive scalar field which is a coherent superposition of two attractive scalar fields of coefficients c_1 and c_2 is $c = c_1 + c_2$, and the constant C in (27) becomes

$$C = [1 - (c_1 + c_2)^2]^{1/2}; \tag{37}$$

these c_1 and c_2 can be of the same or of opposite sign. An incoherent superposition of two scalar fields of the same class occurs similarly to the superposition of an electrostatic potential (repulsive) and a scalar magnetostatic potential (also repulsive) originated by magnetic monopoles; neither do the corresponding sources add up, nor has the mathematical addition of the two scalar fields any physical sense; the constant C in (27) becomes in this case

$$C = [1 \mp (c_1^2 + c_2^2)]^{1/2}. \tag{38}$$

Appendix

Consider the energy–momentum tensor

$$T^{\mu\nu} = \rho u^\mu u^\nu + S^{\mu\nu}, \tag{A.1}$$

where ρ is a mass density, $u^\mu = dx^\mu/ds$ is the corresponding four-velocity vector field, and $S^{\mu\nu}$ is the energy–momentum tensor associated with a scalar field S according to

$$S_{\mu\nu} = \pm(S_{,\mu}S_{,\nu} - g_{\mu\nu}S^{;\alpha}S_{,\alpha}/2)/4\pi. \tag{A.2}$$

The conservation equation $T^{\mu\nu}{}_{;\nu} = 0$ gives

$$\rho w^{\mu} = \mp S^{;\mu} S^{;\alpha}_{;\alpha}/4\pi, \tag{A.3}$$

where w^{μ} is the four-acceleration $u^{\mu}_{;\nu}u^{\nu}$.

In a locally Minkowskian coordinate system with signature -2 we get

$$\rho w^{i} = -(\mp S^{;\alpha}_{;\alpha}/4\pi)S_{,i}; \tag{A.4}$$

the right-hand side is then the ith component of the density of force (of non-gravitational nature) which is acting upon the dust; the expression of this force contains a scalar quantity times the gradient $S_{,i}$ of the scalar field. Following the usual definition of the value of a source as the negative ratio of the force experienced by that source by the gradient of the (potential) field, we see that the density σ of scalar source is given by

$$\sigma = \mp S^{;\alpha}_{;\alpha}/4\pi, \tag{A.5}$$

or in static systems

$$\Delta S = \pm 4\pi\sigma. \tag{A.6}$$

By analogy with the equation for the attractive Newtonian potential $\Delta V = 4\pi\rho$ and that for the repulsive Coulomb potential $\Delta\phi = -4\pi\lambda$ we are now able to relate the upper and lower signs in (A.6) with the attractive and repulsive scalar fields, respectively (see § 2).

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